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Derivation of the Korteweg–deVries Equation for an Operator with a Mixed Spectrum *

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We consider the Hill's equation

$$y'' + [\lambda - q(t)]\, y = 0 \tag{1}$$

with a π-periodic potential function $q(t)$. The discriminant of (1) is given by $\Delta(\lambda) = y_1(\pi) + y_2'(\pi)$ where y_1 and y_2 are the solutions of (1) which satisfy $y_1(0) = y_2'(0) = 1$ and $y_1'(0) = y_2(0) = 0$. The values of λ for which $\Delta(\lambda) = \pm 2$ will be denoted by $\{\lambda_i\}_0^\infty$ and it is known that

$$\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < \cdots < \lambda_{2k-1} \leqslant \lambda_{2k} < \cdots.$$

Intervals of the form $(\lambda_{2k-1}, \lambda_{2k})$ and $(-\infty, \lambda_0)$ are called instability intervals. Goldberg [1] and Lax [6, 7] showed that all but n of the finite instability intervals of (1) will vanish if and only if the potential function $q(t)$ satisfies the $2n$th-order Korteweg–deVries equation

$$\sum_{k=0}^{n+1} C_k S_k(t) = 0, \tag{2}$$

where C_k is a constant and $S_k(t)$ is generated by

$$\begin{aligned} S_0 &= 1, \\ S_{k+1}' &= -\tfrac{1}{4} S_k''' + q S_k' + \tfrac{1}{2} q' S_k. \end{aligned} \tag{3}$$

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Hochstadt and Goldberg [5] have recently shown that when n finite instability intervals of (1) fail to vanish, specification of the endpoints of these nonvanishing intervals $\{\lambda_k\}_0^n$ together with the point spectrum $\{\mu_k\}_1^n$ of

$$L_y = -y'' + q(t)\, y \tag{4}$$

$$y(0) = 0, \qquad y'(0) = -1 \tag{5}$$

leads to a unique construction of $q(t)$. In the same article it was also shown that $q(t)$ can be constructed uniquely if $\{\upsilon_k\}_1^n$, the point spectrum of (4) with boundary conditions

$$y(0) = \cos\alpha, \qquad y'(0) = -\sin\alpha, \qquad \alpha \neq \pi/2, \tag{6}$$

is specified together with $\{\lambda_k\}_0^{2n}$. In this article we will show that the potentials constructed in [5] must satisfy the appropriate Korteweg–deVries equation (2) whether or not $\alpha = \pi/2$. We also offer an additional derivation of (2) which clearly identifies the constants C_k in terms of the λ_k.

To achieve the above we consider the related equation

$$u'' + [\lambda - q(t+\tau)]\, u = 0. \tag{7}$$

By $u_1(t, \tau)$ and $u_2(t, \tau)$ we denote the solutions of (7) which satisfy $u_1(0, \tau) = u_2'(0, \tau) = 1$ and $u_1'(0, \tau) = u_2(0, \tau) = 0$. The simple Hill's spectrum $\{\lambda_i\}$ and the discriminant $\Delta(\lambda)$ are unchanged by shifts in τ [3], but the point spectra $\{\mu_i\}_1^n$ and $\{\upsilon_i\}_0^n$ are τ dependent [4]. In [5] it was shown that when $\alpha = \pi/2$

$$u_2(\pi, \tau) = \frac{a_n(\lambda, \tau)}{2} \sqrt{\frac{4 - \Delta^2(\lambda)}{b_{2n+1}(\lambda)}}, \tag{8}$$

where

$$a_n(\lambda, \tau) = \prod_{i=1}^{n} (\lambda - \mu_i(\tau))$$

and

$$b_{2n+1}(\lambda) = \prod_{i=0}^{2n} (\lambda - \lambda_i).$$

When $\alpha \neq \pi/2$, it can be similarly shown that

$$2u_2(\pi, \tau) \tan\alpha + \frac{\partial}{\partial\tau} u_2(\pi, \tau) = e_n(\lambda, \tau) \sqrt{\frac{4 - \Delta^2(\lambda)}{b_{2n+1}(\lambda)}}, \tag{9}$$

where $e_n(\lambda, \tau)$ depends on $\{v_i\}_0^n$. By inserting (8) into (9) we obtain

$$e_n(\lambda, \tau) = a_n(\lambda, \tau) \tan \alpha + \frac{1}{2} \frac{\partial}{\partial \tau} a_n(\lambda, \tau)$$

which relates $\{\mu_i\}_1^n$ to $\{v_i\}_0^n$ and which leads us to conclude that $e_n(\lambda, \tau)$ is a polynomial in λ of degree n when $\alpha \neq 0$ and $\deg(e_n) = n - 1$ when $\alpha = 0$.

Now $\Delta(\lambda)$, an entire function of λ of order $\frac{1}{2}$, has the following asymptotic representation for real λ [3]:

$$\Delta(\lambda) = 2 \cos \sqrt{\lambda}\, \pi + 2 \cos \sqrt{\lambda}\, \pi \sum_{k=2}^{\infty} \frac{w_k}{\lambda^k} + \frac{2 \sin \sqrt{\lambda}\, \pi}{\sqrt{\lambda}} \sum_{k=1}^{\infty} \frac{v_k}{\lambda^k}$$

so that

$$\begin{aligned} 4 - \Delta^2(\lambda) = {} & 4 \sin^2 \sqrt{\lambda}\, \pi \left[1 + W^2 + 2W - \frac{V^2}{\lambda} \right] \\ & - 8V(W+1) \frac{\cos \sqrt{\lambda}\, \pi \sin \sqrt{\lambda}\, \pi}{\sqrt{\lambda}} - 4(W^2 + 2W), \end{aligned} \tag{10}$$

where

$$W = \sum_{k=2}^{\infty} \frac{wk}{\lambda^k} \qquad \text{and} \qquad V = \sum_{k=1}^{\infty} \frac{v_k}{\lambda^k}.$$

Therefore

$$\sqrt{\frac{4 - \Delta^2(\lambda)}{b_{2n+1}(\lambda)}} = \left(\frac{\begin{aligned} & 4 \sin^2 \sqrt{\lambda}\, \pi (1 + W^2 + 2W - V^2/\lambda) - 8V(W+1) \\ & \qquad \times \cos \sqrt{\lambda}\, \pi \sin \sqrt{\lambda}\, \pi - 4(W^2 + 2W) \end{aligned}}{(\lambda - \lambda_0)(\lambda - \lambda_1) \cdots (\lambda - \lambda_{2n})} \right)^{1/2}. \tag{11}$$

Furthermore, it is known [1, 2] that

$$u_2(\pm\pi, \tau) = \pm \frac{\sin \sqrt{\lambda}\, \pi}{\sqrt{\lambda}} \sum_{k=0}^{\infty} \frac{S_k(\tau)}{\lambda^k} + \cos \sqrt{\lambda}\, \pi \sum_{k=2}^{\infty} \frac{T_k^{\pm}(\tau)}{\lambda^k}, \tag{12}$$

where the $S_k(\tau)$ satisfy (3), so that

$$\begin{aligned} & 2u_2(\pi, \tau) \tan \alpha + \frac{\partial}{\partial \tau} u_2(\pi, \tau) \\ & \quad = \frac{\sin \sqrt{\lambda}\, \pi}{\sqrt{\lambda}} \sum_{k=0}^{\infty} \frac{R_k(\tau)}{\lambda^k} + \cos \sqrt{\lambda}\, \pi \sum_{k=2}^{\infty} \frac{\tilde{T}_k(\tau)}{\lambda^k}, \end{aligned} \tag{13}$$

where

$$R_k(\tau) = 2S_k(\tau)\tan\alpha + S_k'(\tau). \tag{14}$$

The polynomial $e_n(\lambda, \tau)$ can be expressed in the following form.

$$e_n(\lambda, \tau) = \lambda^n \sum_{k=0}^{n} \frac{E_k(\tau)}{\lambda^k} \qquad (E_0 = 0), \text{ when } \alpha = 0). \tag{15}$$

Assuming $\alpha \neq \pi/2$, we substitude (11), (13) and (15) into (9) to obtain

$$\sum_{k=0}^{\infty} \frac{E_k(\tau)}{\lambda^k} \times \left(\frac{4\sin^2\sqrt{\lambda}\,\pi(1 + W^2 + 2W + V^2/\lambda) - 8V(W+1) \times \cos\sqrt{\lambda}\,\pi \sin\sqrt{\lambda}\,\pi - 4(W^2 + 2W)}{\lambda\left(1 - \frac{\lambda_0}{\lambda}\right)\left(1 - \frac{\lambda_1}{\lambda}\right)\cdots\left(1 - \frac{\lambda_{2n}}{\lambda}\right)} \right)^{1/2}$$
$$= \frac{\sin\sqrt{\lambda}\,\pi}{\sqrt{\lambda}} \sum_{k=0}^{\infty} \frac{R_k(\tau)}{\lambda^k} + \cos\sqrt{\lambda}\,\pi \sum_{k=2}^{\infty} \frac{\tilde{T}_k(\tau)}{\lambda^k}. \tag{16}$$

Squaring both sides of (16) and comparing common terms in the asymptotic series, we obtain

$$\sum_{k=0}^{\infty} \frac{R_k(\tau)}{\lambda^k} = \sum_{k=0}^{\infty} \frac{2E_k(\tau)}{\lambda^k} \sqrt{\frac{1 + W^2 + 2W - V^2/\lambda}{\left(1 - \frac{\lambda_0}{\lambda}\right)\left(1 - \frac{\lambda_1}{\lambda}\right)\cdots\left(1 - \frac{\lambda_{2n}}{\lambda}\right)}}. \tag{17}$$

Noting that the terms under the radical are independent of τ and depend only on $\{\lambda_i\}_0^{2n}$, (17) is rewritten as

$$\sum_{k=0}^{\infty} \frac{G_k}{\lambda^k} \sum_{k=0}^{\infty} \frac{R_k(\tau)}{\lambda^k} = \sum_{k=0}^{\infty} \frac{E_k(\tau)}{\lambda^k}. \tag{18}$$

From the coefficient of $\lambda^{-(n+1)}$ we get

$$\sum_{k=0}^{n+1} G_{n+1-k} R_k(\tau) = 0. \tag{19}$$

We now let $C_k = G_{n+1-k}$ in (19) and use (14) to obtain

$$2\tan\alpha \sum_{k=0}^{n+1} C_k S_k(\tau) + \frac{d}{d\tau} \sum_{k=0}^{n+1} C_k S_k(\tau) = 0,$$

which implies that

$$\sum_{k=0}^{n+1} C_k S_k(\tau) = De^{-2\tau \tan \alpha}, \qquad D \text{ constant.}$$

In [2], it was shown that $u_2(\pi, \tau)$ is periodic with period π, and from (12) it is clear that $S_k(\tau)$ is also periodic with period π, so that D must vanish and we can now conclude that when $\alpha \neq \pi/2$,

$$\sum_{k=0}^{n+1} C_k S_k(\tau) = 0. \tag{20}$$

Since $D = 0$, (20) is independent of the point spectrum $\{\upsilon_k\}_0^n$. Thus the Hill's spectrum $\{\lambda_k\}_0^{2n}$ alone relates the potential q to the Kortweg–deVries equation.

Equation (20) corresponds to the result announced in [1] where the constants C_k were not identified. To show that (20) is identical with (2), we must show that their coefficients agree. To accomplish this, we let $g(x, \varepsilon)$ be the Green's function which satisfies

$$\begin{gathered} g'' + [\lambda - q(x+\tau)]\, g = \delta(x-\varepsilon), \qquad 0 < \varepsilon < 2\pi \\ g(0) = g(2\pi), \qquad g'(0) = g'(2\pi). \end{gathered} \tag{21}$$

A standard calculation [2] shows that

$$g(x) \equiv g(x, 0) = \frac{u_2(x, \tau) - u_2(x - 2\pi, \tau)}{4 - \Delta^2(\lambda)}. \tag{22}$$

From our hypothesis, the only simple zeros of $4 - \Delta^2(\lambda)$ are $\lambda_0, \lambda_1, \ldots, \lambda_{2n}$, all others are double so that

$$4 - \Delta^2(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_1) \cdots (\lambda - \lambda_{2n}) f^2(\lambda). \tag{23}$$

From (12), (22) and (23), we have

$$\begin{aligned} g(\pi) &= \frac{u_2(\pi, \tau) - u_2(-\pi, \tau)}{4 - \Delta^2(\lambda)} \\ &= \frac{\dfrac{2 \sin \sqrt{\lambda}\, \pi}{\sqrt{\lambda}} \displaystyle\sum_{k=0}^{\infty} \frac{S_k(\tau)}{\lambda^k} + 2 \cos \sqrt{\lambda}\, \pi \sum_{k=0}^{\infty} \frac{T_k^*}{\lambda^k}}{(\lambda - \lambda_0)(\lambda - \lambda_1) \cdots (\lambda - \lambda_{2n}) f^2(\lambda)} \end{aligned}$$

or

$$(\lambda-\lambda_0)(\lambda-\lambda_1)\cdots(\lambda-\lambda_{2n})\,f(\lambda)\,g(\pi)$$
$$=\frac{\dfrac{2\sin\sqrt{\lambda}\,\pi}{\sqrt{\lambda}}\displaystyle\sum_{k=0}^{\infty}\frac{S_k(\tau)}{\lambda^k}+2\cos\sqrt{\lambda}\,\pi\sum_{k=0}^{\infty}\frac{T_k^*}{\lambda^k}}{f(\lambda)}. \tag{24}$$

Now $g(x)$ must be a meromorphic function of λ and since boundary value problem (21) is self-adjoint, g can only have simple poles. Therefore, each side of (24) must be an entire function. Using (23), we rewrite $f(\lambda)$ as

$$f(\lambda)=\sqrt{\frac{4-\Delta^2(\lambda)}{(\lambda-\lambda_0)(\lambda-\lambda_1)\cdots(\lambda-\lambda_{2n})}}=\sqrt{\frac{4-\Delta^2(\lambda)}{b_{2n+1}(\lambda)}}. \tag{25}$$

From (10), we see that

$$f(\lambda)\approx\frac{2\sin\sqrt{\lambda}\,\pi}{\lambda^{n+1/2}}\qquad\text{as}\quad\lambda\to\infty.$$

It follows that the right-hand side of (24) is $O(\lambda^n)$, so that by Liouville's theorem, it must be a polynomial of degree n in λ, say,

$$p_n(\lambda,\tau)\equiv\lambda^n\sum_{k=0}^{n}\frac{P_k(\tau)}{\lambda^k}. \tag{26}$$

From (24) and (25), we now have

$$\frac{2\sin\sqrt{\lambda}\,\pi}{\sqrt{\lambda}}\sum_{k=0}^{\infty}\frac{S_k(\tau)}{\lambda^k}+2\cos\sqrt{\lambda}\,\pi\sum_{k=1}^{\infty}\frac{T_k^*(\tau)}{\lambda^k}$$
$$=p_n(\lambda,\tau)\sqrt{\frac{4-\Delta^2(\lambda)}{b_{2n+1}(\lambda)}}. \tag{27}$$

Using (26) and (11) in the right-hand side and following (16) through (19), we conclude that

$$\sum_{k=0}^{n+1}G_{n+1-k}S_k(\tau)=0$$

must be satisfied. This clearly is identical with (20).

When $\alpha = \pi/2$, we substitute (12) into (8) to obtain

$$\frac{2 \sin \sqrt{\lambda}\, \pi}{\sqrt{\lambda}} \sum_{k=0}^{\infty} \frac{S_k(\tau)}{\lambda^k} + 2 \cos \sqrt{\lambda}\, \pi \sum_{k=1}^{\infty} \frac{T_k^+(\tau)}{\lambda^k}$$
$$= a_n(\lambda, \tau) \sqrt{\frac{4 - \Delta^2(\lambda)}{b_{2n+1}(\lambda)}}. \tag{28}$$

Using (11) for the radical and squaring both sides of (27) and of (28), a comparison of coefficients of sine terms yields

$$a_n(\lambda, \tau) = p_n(\lambda, \tau). \tag{29}$$

We also note that since $2T_k^*(\tau) = T_k^+(\tau) - T_k^-(\tau)$, a comparison of cosine terms yields the by-product

$$T_k^+(\tau) = -T_k^-(\tau).$$

It is now obvious that (28) is equivalent to (27) and leads us to (20) in the same manner. Here, too, we see that (20) is independent of the point spectrum $\{\mu_k\}_1^n$ and only depends on the Hill's spectrum $\{\lambda_k\}_0^{2n}$. By matching coefficients of λ^k in (29), one can recover the trace relations referred to in [5] and [8].

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